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On a uniformly valid model for surface wave interaction

By YEHUDA AGNON
Coastal and Marine Engineering Research Institute, Department of Civil Engineering, Technion, Haifa 32000, Israel

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Nonlinear interaction of surface wave trains is studied. Zakharov's kernel is extended to include the vicinity of trio resonance. The forced wave amplitude and the wave velocity changes are then first order rather than second order. The model is applied to remove near-resonance singularities in expressions for the change of speed of one wave train in the presence of another. New results for Wilton ripples and the drift current and setdown in shallow water waves are readily derived. The ideas are applied to the derivation of forced waves in the vicinity of quartet and quintet resonance in an evolving wave field.

1. Introduction

One or several waves may have a nonlinear interaction which modifies their velocities and gives rise to sum and difference wavenumber waves. We choose for the present discussion gravity–capillary free surface waves.

In the case of a single wave, the change in frequency is referred to as Stokes' correction and the sum and difference wavenumber waves are the second harmonic and the drift current (and wave setdown), respectively. The expressions for the drift current and setdown (Longuet-Higgins & Stewart, 1962) become singular when the wavelength is large compared to the water depth and the group velocity approaches the long-wave velocity $(gh)^{\frac{1}{2}}$. This is long-wave resonance.

Resonance with the second harmonic exists in Wilton ripples. It occurs when the phase velocity of the first harmonic is close to that of the second harmonic. Pierson & Fife (1961) have found a bounded solution that exhibits changes in the wave frequencies. These result in detuning of the resonance.

The interaction of two wave trains was studied by Longuet-Higgins & Phillips (1962) and by Hogan, Gruman & Stiassnie (1988, referred to herein as HGS). They found a second-order change in phase speed induced by each wave train on the other. HGS used the Zakharov equation to derive these changes for gravity–capillary waves. They encountered singularities when the two waves were members of a resonant trio.

In the present paper, we study the interaction of two waves, extending the kernel of the Zakharov (1968) equation so that it holds in the vicinity of trio resonance. When the two primary waves are members of a nearly resonant trio, they induce a first-order forced wave rather than a second-order one. In §2, we study the trio interaction of the two primary waves and the forced wave. We obtain a quadratic equation for the forced wave amplitude. The primary waves are found to have frequency shifts that are first order, hence the resonance is detuned. In §3, we derive a kernel for quartet interaction that incorporates the detuning effect. The result is an expression for the frequency shift of a wave train in the presence of another which is uniformly valid, at resonance and away from it. The special case of Wilton ripples (progressive and standing) is studied in §4, giving a uniformly valid result. Section 5 is a treatment of long-wave resonance, where the setdown and drift current become first order. It is wave–mean-flow interaction. The mean flow is derived in an independent manner. Then the similarities to the results of §3 are discussed. In §6, we suggest application of these ideas to the derivation of forced waves in the vicinity of resonance, where the assumption of scale separation used in perturbation expansions may not hold.

2. Trio interaction solution

The interaction of surface waves is most conveniently studied in terms of the cannonical variables $b(k,t)$ defined by Zakharov (1968):

$$\frac{1}{\pi}b(\boldsymbol{k},t)=\left[\frac{\omega}{2|\boldsymbol{k}|}\right]^{\frac{1}{2}}\hat{\xi}(\boldsymbol{k},t)+\mathrm{i}\left[\frac{|\boldsymbol{k}|}{2\omega}\right]^{\frac{1}{2}}\hat{\phi}^{\mathrm{s}}(\boldsymbol{k},t), \tag{2.1}$$

where $\hat{\xi}$ and $\hat{\phi}^{\mathrm{s}}$ are the spatial Fourier transforms of the free-surface elevation, ξ, and the velocity potential at the free surface, ϕ^{s}. $\boldsymbol{k}$ is the wavenumber vector, ω the frequency from the linear dispersion relation and t is time.

In §§2 and 3, we study the interaction of the two free waves and a forced wave which may be in near resonance with the free waves.

Since we wish to account for trio resonance, we write the Zakharov equation in a form that includes trio and quartet interaction:

$$\begin{aligned}\frac{\partial}{\partial t}b(\boldsymbol{k}_0,t)+\mathrm{i}\omega(\boldsymbol{k}_0)\,b(\boldsymbol{k}_0,t) &= -\mathrm{i}\iint_{-\infty}^{\infty}V^{(-)}_{0,1,2}\,b(\boldsymbol{k}_1,t)\,b(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}_0-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\\ &\quad -\mathrm{i}\iint_{-\infty}^{\infty}2V^{(-)}_{2,1,0}\,b^*(\boldsymbol{k}_1,t)\,b(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}_0+\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\\ &\quad -\mathrm{i}\iiint_{-\infty}^{\infty}W_{0,1,2,3}\,b^*(\boldsymbol{k}_1,t)\,b(\boldsymbol{k}_2,t)\,b(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}_0+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3.\end{aligned} \tag{2.2}$$

The interaction coefficients $V^{(-)}$ and W are given in HGS and in the Appendix of this paper. They are functions of $\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2$ and ω_0, ω_1, ω_2. The asterisk denotes the complex conjugate. The Zakharov equation describes the time evolution in water of constant depth. No spatial modulation is considered. In (2.2) we have included only interactions that can be nearly resonant. We start by studying the case of nearly resonant trios, focusing on the leading-order trio interaction. In §3, a more complete treatment will be given.

Consider a primary wave field made up of two waves, and include the forced sum (+) or difference (−) wave. b may be written in terms of amplitudes B_1, B_2 and B_3:

$$\begin{aligned}b(\boldsymbol{k},t) &= b_1(t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1)+b_2(t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_2)+b_3(t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_3)\\ &= B_1(t_1)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{e}^{-\mathrm{i}\omega_1 t}+B_2(t_1)\,\delta(\boldsymbol{k}-\boldsymbol{k}_2)\,\mathrm{e}^{-\mathrm{i}\omega_2 t}+B_3(t_1)\,\delta(\boldsymbol{k}-\boldsymbol{k}_3)\,\mathrm{e}^{-\mathrm{i}(\omega_1\pm\omega_2)t},\end{aligned} \tag{2.3}$$

where $\boldsymbol{k}_3=\boldsymbol{k}_1\pm\boldsymbol{k}_2$ is the wavenumber of the forced wave and $\omega_1\pm\omega_2$ its frequency, $t_1\equiv\epsilon t$, and ϵ is the wave steepness. Since the nonlinear term is quadratic in ϵ, trio interaction occurs on the slow timescale t_1. This is why the two primary wave amplitudes B_1 and B_2 were taken to be functions of t_1, at most. B_3 represents a locked wave (again, only the term that can lead to resonance, either sum or difference, is included). B_1 and B_2 are $O(\epsilon)$. Away from resonance B_3 is $O(\epsilon^2)$.

Denote by μ the resonance detuning:

$$\mu = \omega_3 - (\omega_1 \pm \omega_2); \tag{2.4}$$

$\omega_3 = \omega(\boldsymbol{k}_3)$ is the free wave frequency of $\boldsymbol{k}_3$. Resonance occurs when $\mu = O(\epsilon)$. In that case, we shall see that $B_3 = O(\epsilon)$.

We look for steady solutions with amplitudes A_j and frequencies Ω_j:

$$B_j(t_1) = A_j \exp(-\mathrm{i}(\Omega_j - \omega_j)t); \quad j = 1,2,3; \quad A_j \quad \text{real constants}, \tag{2.5}$$

where no energy is exchanged among the wave modes.

From (2.2) we get (including only trio interaction)

$$(\omega_j - \Omega_j)A_j = -2VA_m A_l; \quad j = 1,2,3; \quad j+m+l = 6, \tag{2.6}$$

where

$$V = \begin{cases} V^{(-)}_{3,1,2} & \text{if} \quad \boldsymbol{k}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2 \\ V^{(-)}_{1,2,3} & \text{if} \quad \boldsymbol{k}_3 = \boldsymbol{k}_1 - \boldsymbol{k}_2, \end{cases} \quad \text{and} \quad \Omega_1 \pm \Omega_2 = \Omega_3. \tag{2.7}$$

Let us assume that A_1 and A_2 are specified. Eliminating Ω_1, Ω_2 and Ω_3 we get from (2.6) and (2.7) a quadratic equation for A_3:

$$\left(\frac{A_2}{A_1} \pm \frac{A_1}{A_2}\right) A_3^2 - \frac{\mu}{2V} A_3 - A_1 A_2 = 0. \tag{2.8}$$

If instead we eliminate A_3, we get

$$(\omega_1 - \Omega_1)A_1 = -2VA_2A_3 = \frac{(2V)^2 A_1 A_2^2}{\omega_3 - \Omega_3} = \frac{(2V)^2 A_1 A_2^2}{\omega_3 - (\Omega_1 \pm \Omega_2)}. \tag{2.9}$$

Note that the right-hand side of (2.9) represents a contribution to the quartet interaction of waves (1, 1, 2, 2), through trio interaction of wave 3 (which is induced by waves 1 and 2) with waves 1 and 2. Hence the quadratic appearance of V.

From (2.8), we see that at resonance, A_3 and $\Omega_j - \omega_j, j = 1,\ 2,\ 3$ are all $O(\epsilon)$. This is in contrast with the non-resonant case, $\mu \gg \epsilon$, when $\Omega_3 - \omega_3 \approx \mu$ and $A_3 \approx -2VA_1A_2/\mu = O(\epsilon^2)$, while $\Omega_j - \omega_j, j = 1,2$ are $O(\epsilon^2)$. That is, the forced wave has a negligible effect on the frequencies of the primary waves. The frequency shifts $\Omega_j - \omega_j$ represent detuning of the resonance. In the next section we shall see that result (2.9) appears in the corrected form of the quartet interaction kernel, a form which accounts for trio resonance.

3. Derivation of a uniformly valid kernel for quartet interaction

Stiassnie & Shemer (1987) have developed a scheme for computing class I (quartet) and class II (quintet) interaction, based on their modification of the Zakharov equation (Stiassnie & Shemer 1984). The scheme is based on separating the spectral components into free waves and forced waves:

$$b(\boldsymbol{k},t) = [B(\boldsymbol{k},t_2) + B'(\boldsymbol{k},t,t_2) + B''(\boldsymbol{k},t,t_2) + \ldots]\,\mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{k})t}, \tag{3.1}$$

where $t_2 \equiv \epsilon^2 t$, and

$$B = O(\epsilon), \quad B' = O(\epsilon^2), \quad B'' = O(\epsilon^3).$$

If one excludes trio resonance conditions, B depends only on t_2.

The interaction equation in discretized form is

$$\frac{\partial}{\partial t}\tilde{B}_0 = -\mathrm{i} \sum_{k_1} \sum_{k_2} \sum_{k_3} T_{0,1,2,3} B_1^* B_2 B_3\, \delta(\boldsymbol{k}_0 + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\,\mathrm{e}^{\mathrm{i}\mu_1 t}, \tag{3.2}$$

where μ_1 is the detuning:

$$\mu_1 \equiv \omega + \omega_1 - \omega_2 - \omega_3; \quad \text{where} \quad \omega = \omega(\boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}_1).$$

$T_{0,1,2,3} = W_{0,1,2,3} +$ quadratic terms in V (see the Appendix) is a modified quartet interaction coefficient which includes trio interaction of any two free waves, with locked waves which are, in turn, forced by trio interaction of any two free waves, hence the quadratic terms in V. $\tilde{B}_0$ may stand for $B(\boldsymbol{k}_0)$ or $B''(\boldsymbol{k}_0)$, if we let μ_1 be $O(\epsilon^2)$ or $O(1)$, respectively.

HGS studied the interaction of two uniform wave trains. Looking for steady solutions to (3.2) of the form

$$B_j(t_2) = A_j \exp\left[-\mathrm{i}(\Omega_j - \omega_j)t\right]; \quad j = 1, 2, \tag{3.3}$$

they immediately find for the shifted frequencies Ω_j the following generalized Stokes' corrections:

$$\left.\begin{aligned} \Omega_1 - \omega_1 &= T_1 A_1^2 + 2T_{1,2} A_2^2, \\ \Omega_2 - \omega_2 &= T_2 A_2^2 + 2T_{1,2} A_1^2, \end{aligned}\right\} \tag{3.4}$$

where $T_1 \equiv T_{1,1,1,1}$; $T_2 \equiv T_{2,2,2,2}$; $T_{1,2} \equiv T_{1,2,1,2}$.

We can relate these expressions to our results of §2. Only trio interactions were discussed there. The corresponding term in HGS neglects the frequency shifts and uses the linear ω_j in the denominator. The terms in (3.4) that contain $T_{1,2}$ include the trio interaction of waves 1 and 2 with wave 3, which is generated by trio interaction with waves 1 and 2. This appears in $T_{1,2}$ as the term

$$2V^2/[\omega_3 - (\omega_1 \pm \omega_2)]. \tag{3.5}$$

This term corresponds to the coefficient of $2A_1 A_2^2$ in (2.9), which reduces to the form (3.5) away from resonance. Near resonance we need to account for the frequency shifts and use the shifted Ω_j as was done in (2.9). Hence, we need to replace the term (3.5) in the expression for $T_{1,2}$ (see Appendix) by

$$2V^2/[\omega_3 - (\Omega_1 \pm \Omega_2)]. \tag{3.6}$$

We obtain a kernel $\hat{T}_{1,2}$ that is valid near trio resonance as well. This replacement is significant only when μ, the denominator of (3.5) is small. Using the new form of $T_{1,2}$, we obtain

$$\left.\begin{aligned} \Omega_1 - \omega_1 &= T_1 A_1^2 + 2\left[\frac{-2V^2}{\omega_3 - \Omega_3} + T'_{1,2}\right] A_2^2 = T_1 A_1^2 + 2\hat{T}_{1,2} A_2^2, \\ \Omega_2 - \omega_2 &= T_2 A_2^2 + 2\left[\frac{-2V^2}{\omega_3 - \Omega_3} + T'_{1,2}\right] A_1^2 = T_2 A_2^2 + 2\hat{T}_{1,2} A_1^2, \end{aligned}\right\} \tag{3.7}$$

where

$$T'_{1,2} \equiv T_{1,2} + \frac{2V^2}{\omega_3 - (\omega_1 \pm \omega_2)}; \quad \hat{T}_{1,2} = T'_{1,2} - \frac{2V^2}{\omega_3 - (\Omega_1 \pm \Omega_2)}.$$

We shall see that $T'_{1,2}$ is the kernel with the singular behaviour removed. T_1, T_2 and $T'_{1,2}$ are $O(1)$, while $-2V^2/(\omega_3 - \Omega_3)$ is $O(\mu^{-1})$. A_1^2 and A_2^2 are $O(\epsilon^2)$. Thus, away from resonance, $\Omega_1 - \omega_1$ (and $\Omega_2 - \omega_2$) are $O(\epsilon^2)$. Near resonance, where μ decreases to $O(\epsilon)$, $\Omega_1 - \omega_1$ (and $\Omega_2 - \omega_2$) increase to $O(\epsilon)$. Keeping only the leading term in the square brackets of (3.7) then leads to the results of §2 (equation (2.9)).

In order to solve for A_3 and Ω_j, it is convenient to define

$$y \equiv \Omega_1 - \omega_1 \pm (\Omega_2 - \omega_2) = \mu + (\Omega_3 - \omega_3)$$

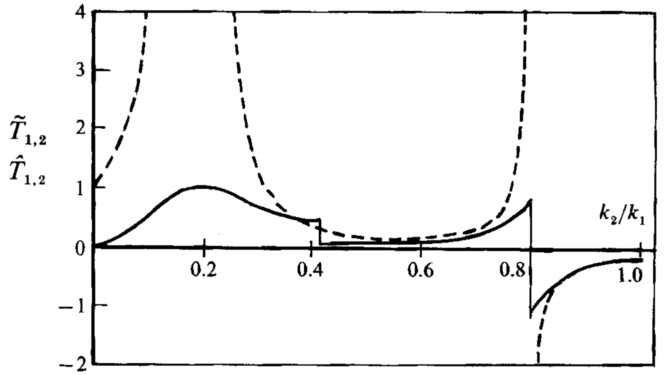

FIGURE 1. The dependence of the normalized interaction coefficient on k_2/k_1. $\tilde{T}_{1,2}$ (---) is compared with $4\pi^2\hat{T}_{1,2}/k_1k_2^2$ (——). $a_1 = a_2 = 0.01$ cm. $2\pi/k_1 = 1$ cm.

and add up (or subtract, for difference interaction) (3.7). We get a quadratic equation in y, the free-wave frequency shift:

$$y^2-\beta y-\gamma = 0, \tag{3.8}$$

where

$$\left.\begin{aligned}\beta &= \mu+T_1A_1^2\pm T_2A_2^2+2T'_{1,2}(A_2^2\pm A_1^2) = O(\mu,\epsilon^2),\\ \gamma &= 4V^2(A_2^2\pm A_1^2)-\mu[T_1A_1^2\pm T_2A_2^2+2T'_{1,2}(A_2^2\pm A_1^2)] = O(\epsilon^2),\end{aligned}\right\} \tag{3.9}$$

are known. We see that at resonance, $\mu = O(\epsilon)$, we get $y = O(\epsilon)$. When μ decreases to $O(\epsilon)$, the forced wave A_3 increases from $O(\epsilon^2)$ to $O(\epsilon)$ but remains bounded. Note that even when μ is zero, the shift y is not. A_3 is given by (2.6). In the non-resonant case the primary waves have a frequency shift which is $O(\epsilon^2)$. It is $O(\epsilon)$ when $\mu = O(\epsilon)$.

In figure 1 we show the dependence of $\tilde{T}_{1,2} \equiv 4\pi^2 T_{1,2}/k_1k_2^2$ on k_2/k_1 for the case presented in figure 5 of HGS: $2\pi/k_1 = 1$ cm. The waves are taken to be parallel. A region which includes two resonances is shown. The singular behaviour of $\tilde{T}_{1,2}$ (dashed line) is compared to the expression in the present theory: $\hat{T}_{1,2}$ normalized in the same way, computed with $a_1 = a_2 = 0.01$ cm. a_1, a_2 are the wave amplitudes given by $a_i = (2k_i/\omega_i)^{\frac{1}{2}}A_i/2\pi$ ($i = 1, 2$). The finite height of the resonance peak is due to the finite steepness of the waves. For infinitesimal waves, the resonance would be infinite (there would be no detuning). Indeed, computations show that steeper waves result in relatively smaller resonance (when normalized by $A_1A_2 = O(\epsilon^2)$). The discontinuity near $k_2 = 0.4k_1$ is due to resonance of k_2 with its own second harmonic. This resonance is studied in the following section where we apply the theory to the well-known special case of Wilton ripples.

4. Wilton ripples

An interesting special case of trio resonance is Wilton ripples. This is the simplest sum interaction resonance. In this case, there is trio near resonance between a wave and its second harmonic:

$$\mathbf{k}_1 = \mathbf{k}_2; \quad \mathbf{k}_3 = 2\mathbf{k}_1; \quad \omega_3 = 2\omega_1+\mu.$$

In the terminology of HGS, we have a single wave interacting with itself. Looking for a periodic solution, we set

$$\Omega_3 = 2\Omega_1. \tag{4.1}$$

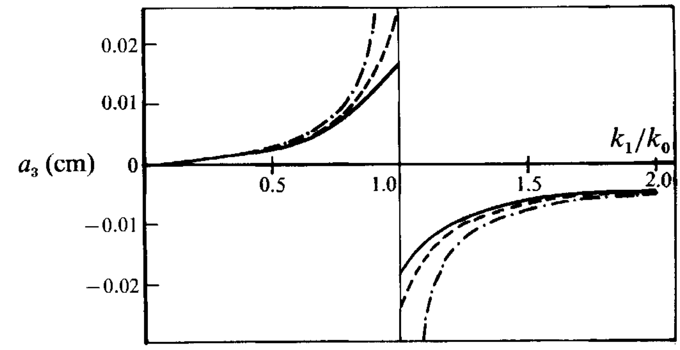

FIGURE 2. The dependence of the second harmonic a_3 (in cm) on k_1/k_0 ($2\pi/k_0 = 2.44$ cm). Shown are Stokes' expansion (-·-·-), Pierson & Fife's result (----), and present theory (——). $a_1 = 0.05$ cm.

Equation (3.7) now takes the form

$$\Omega_1-\omega_1=\left[\frac{-2V^2}{\omega_3-2\Omega_1}+T'_{1,1}\right]A_1^2. \tag{4.2}$$

A_3 is given by

$$(\omega_3-2\Omega_1)A_3=-VA_1^2. \tag{4.3}$$

This time $y=2(\Omega_1-\omega_1)$ and we have

$$y^2-\beta y-\gamma=0, \tag{4.4}$$

where

$$\beta=\mu+2T'_{1,1}A_1^2,\quad \gamma=(8V^2-2\mu T'_{1,1})A_1^2.$$

If we neglect $2T'_{1,1}A_1^2$ (which is $O(A_1^2)=O(\epsilon^2)$) by arguing that it is small compared to μ, which is, say, $O(\epsilon)$, we get from (4.3) and (4.4)

$$(A_3-\mu/8V)^2-\mu^2/64V^2=\tfrac{1}{4}A_1^2. \tag{4.5}$$

Exact second-harmonic resonance occurs at

$$k_0=(g/2S_0)^{\frac{1}{2}};\quad \omega_0=(\tfrac{3}{2})^{\frac{1}{2}}(g^3/2S_0)^{\frac{1}{4}}, \tag{4.6}$$

where g is gravity and S_0 the surface tension coefficient divided by the fluid density. In water, k_0 corresponds to a wavelength of 2.44 cm.

If we set $k_1=k_0+\delta, \delta \ll k_0$, we get

$$\mu\approx g\delta/\omega_0;\quad V\approx(4\pi\sqrt{2})^{-1}k_0^{\frac{3}{2}}\omega_0^{\frac{1}{2}} \tag{4.7}$$

and the surface elevation amplitudes of the two free waves a_1 and a_3

$$A_1/a_1\approx A_3/a_3\approx 2\pi(\omega_0/2k_0)^{\frac{1}{2}}. \tag{4.8}$$

Substituting (4.6)–(4.8) in (4.5) we get

$$(a_3-\tfrac{1}{3}\sqrt{2}\,\delta)^2-\tfrac{2}{9}\delta^2=\tfrac{1}{4}a_1^2, \tag{4.9}$$

which is the result obtained by Pierson & Fife (1961). However, when μ tends to zero, we may no longer neglect $2T'_{1,1}A_1^2$ and (4.4) should be used.

The solution of (4.4) gives a uniformly valid result that exhibits a smooth transition between the regimes near resonance and away from it. In figure 2 we show the dependence of a_3 on μ. The singular behaviour from a Stokes' wave expansion given by (4.3) with Ω_1 replaced by ω_1, is compared to the trio interaction solution of

(4.9) due to Pierson & Fife (1961) and to the uniformly valid quartet interaction solution obtained from (4.4), which bridges the regimes near resonance and away from it. a_1 is 0.05 cm.

The above analysis can be easily extended to trios of standing waves, as well as other problems where resonance is possible. This concludes our analysis of Wilton ripples. We turn to another important special case: long-wave resonance.

5. Long-wave resonance

For a single wave train, difference interaction is interaction with the mean flow. The expression for the long wave induced by a short-wave packet was given by Longuet-Higgins & Stewart (1962) in the form

$$\xi_{10} = -\frac{h}{gh-C_g^2}\frac{S}{\rho h}, \tag{5.1}$$

$$U = -\frac{C_g}{gh-C_g^2}\frac{S}{\rho h}, \tag{5.2}$$

where ξ_{10} is the wave setdown and U is the mean current over the whole depth, h,

$$C_g = \frac{C}{2}\left[1+\frac{2kh}{\sinh(2kh)}\right]$$

is the group velocity, and

$$C = [g\tanh(kh)/k]^{\frac{1}{2}}$$

is the phase velocity of the short waves. S, given by

$$S = \left(\frac{2C_g}{C}-\frac{1}{2}\right)E; \quad E = \tfrac{1}{2}\rho g a^2 \tag{5.3}$$

is the radiation stress, where a is the short-wave amplitude. ρ is the fluid density. These are also the expressions for the mean flow and wave setdown given by Whitham (1974). When $kh \gg 1$, ξ_{10} and U are $O(\epsilon^2)$. As kh decreases, $C_g^2 \to gh$ and the expressions (5.1) and (5.2) become singular. This is due to long-wave resonance.

Djordjevic & Redekopp (1977) have obtained evolution equations for the case of long-wave resonance. In their solution, however, the long wave is free (undetermined) for the case of a wave train that is uniform to leading order. We wish to determine the long wave that is induced by the short waves over a timescale that is shorter than the timescale for evolution of the short-wave envelope (which is the timescale in their work). Our treatment is uniformly valid in the vicinity of the short-wave long-wave resonance, and away from resonance.

We will not use the Zakharov formulation. It leads to the same results, but the derivation is more complicated than the direct approach. The reason is that, due to interaction with itself, the long wave will not be harmonic even if the short-wave envelope is. Thus, we shall obtain a partial differential equation for the long-wave potential which is valid for arbitrary short-wave envelopes. In fact, we are studying the interaction of a short-wave group with the current induced by it. The resulting resonance detuning is analogous to that of the trio and quartet interaction models of §§2 and 3.

Let us write the wave potential ϕ and the free surface displacement ξ as

$$\left.\begin{aligned} \phi &= \phi_0(x_1,t_1)+\phi_{11}\exp(-\mathrm{i}\omega t)+\phi_{11}^*\exp(\mathrm{i}\omega t)+O(\epsilon^2),\\ \xi &= \xi_{10}(x_1,t_1)+\xi_{11}\exp(\mathrm{i}\omega t)+\xi_{11}^*\exp(\mathrm{i}\omega t)+O(\epsilon^2), \end{aligned}\right\} \tag{5.4}$$

where

$$\phi_{11} = \frac{ig}{2\omega} a(x_1, t_1) \exp(ikx) \frac{\cosh k(z+h)}{\cosh kh}, \quad \xi_{11} = \tfrac{1}{2} a(x_1, t_1) \exp(ikx). \tag{5.5}$$

$\phi_0 = O(1)$, $\xi_{10} = O(\epsilon)$. $x_1 \equiv \epsilon x$ is a stretched horizontal coordinate, characterizing amplitude variation. The rest position of the free surface is at $z = 0$. Averaging the continuity equation and the Bernoulli equation over the short scales, we get

$$\frac{\partial}{\partial x_1}[(h+\xi_{10})\,\phi_{0_{x_1}} + \xi_{11}\,\phi^*_{11_x} + \xi^*_{11}\,\phi_{11_x}] + \frac{\partial \xi_{10}}{\partial t_1} = O(\epsilon^4),$$

$$i\omega\phi_{11_z}\,\xi^*_{11} - i\omega\phi^*_{11_z}\,\xi_{11} - \phi_{0_{t_1}} = g\xi_{10} + |\phi_{11_z}|^2 + |\phi_{11_x}|^2 + \tfrac{1}{2}(\phi_{0_{x_1}})^2 + O(\epsilon^3).$$

Eliminating ξ_{10} between the two equations we get

$$\phi_{0_{t_1 t_1}} - gh\phi_{0_{x_1 x_1}} = -\left(k^2 - \sigma^2 + 2\frac{\omega k}{C_g}\right)\frac{g^2 |a^2|_{t_1}}{4\omega^2} - \tfrac{1}{2}[(\phi_{0_{x_1}})^2]_{t_1} - (\phi_{0_{x_1}}\,\phi_{0_{t_1}})_{x_1}, \tag{5.6}$$

where $\sigma = k \tanh(kh)$ (e.g. Agnon & Mei 1985. The quadratic terms in ϕ_0 are required since $\phi_{0_{x_1}} = O(\epsilon)$). The propagation of the short-wave envelope, a, in the presence of the drift current $\phi_{0_{x_1}}$ is given by

$$|a^2|_{t_1} + (C_g + \phi_{0_{x_1}})\,|a^2|_{x_1} = 0 \tag{5.7}$$

(e.g. Whitham 1974, Eq. 16.91).

We look for solutions of the form

$$a = a[x_1 - (C_g + \phi_{0_{x_1}})\,t_1]; \quad \phi_0 = \phi_0[x_1 - (C_g + \phi_{0_{x_1}})\,t_1]. \tag{5.8}$$

We get, after integrating (5.6), the following equation:

$$(C_g^2 - \tfrac{1}{2}C_g\,\phi_{0_{x_1}} - gh)\,\phi_{0_{x_1}} = qC_g\,|a^2| + O(\epsilon^3), \tag{5.9}$$

where $$q \equiv (k^2 - \sigma^2 + 2\omega k/C_g)\,g^2/4\omega^2.$$

This is a quadratic equation for $\phi_{0_{x_1}}$:

$$\tfrac{1}{2}(\phi_{0_{x_1}})^2 + \left(\frac{gh - C_g^2}{C_g}\right)\phi_{0_{x_1}} + q\,|a|^2 = 0,$$

and its solution is

$$\phi_{0_{x_1}} = -\frac{gh - C_g^2}{C_g} - \left[\left(\frac{gh - C_g^2}{C_g}\right)^2 - 2q\,|a^2|\right]^{\frac{1}{2}}. \tag{5.10}$$

Thus we can find U, which is given by

$$U = \int_h^{\xi} \phi_x\,\mathrm{d}z = \phi_{0_{x_1}} + \frac{E}{\rho Ch}, \tag{5.11}$$

where $E/\rho Ch$ is the Stokes' drift.

In the range $gh - C_g^2 = O(1)$, (5.10) and (5.11) yield the classical result (5.2). However, when $gh - C_g^2$ is $O(\epsilon)$, we find that

$$\xi_{10} \approx \frac{h}{C_g} U = O(\epsilon). \tag{5.12}$$

Let us relate these results to the previous sections' results. We write for the long-wave frequency $\omega_0 = \omega(k_2 - k_1) = \Delta\omega + \mu$, where $\Delta\omega = \omega_2 - \omega_1$ is the modulation frequency which tends to zero for an almost uniform wave train. Then, the linear long wave speed is

$$\frac{\omega_0}{k_0} = \frac{\Delta\omega+\mu}{\Delta k} \approx C_g + \frac{\mu}{\Delta k},$$

where $\Delta k = k_2 - k_1 = k_0$. The shifted long-wave frequency is related to the shifted short-wave frequencies by $\Omega_0 = \Delta\Omega = \Omega_2 - \Omega_1$.

The presence of the drift current, $\phi_{0_{x_1}}$, induces a Doppler shift

$$\Omega_j - \omega_j = \phi_{0_{x_1}} k_j; \quad j = 1, 2. \tag{5.13}$$

Thus the shifted group velocity is

$$\frac{\Delta\Omega}{\Delta k} = \frac{\Delta\omega}{\Delta k} + \phi_{0_{x_1}} = C_g + \phi_{0_{x_1}}. \tag{5.14}$$

It is also equal to the shifted long-wave speed

$$\frac{\Omega_0}{k_0} = \frac{\Delta\Omega}{\Delta k} = (gh)^{\frac{1}{2}} + O(\mu). \tag{5.15}$$

So the two velocities match. The frequencies of the 'short' waves are shifted due to the interaction with the long wave. The frequency of the long wave, Ω_0, is shifted by the interaction with the short wave and with itself.

The value of $\phi_{0_{x_1}}$ found from (5.10) is real as long as

$$\left(\frac{gh - C_g^2}{C_g}\right)^2 > 2q\,|a|^2. \tag{5.16}$$

This condition may be violated in shallow water. In shallow water we have $(gh - C_g^2)/C_g \approx (gh)^{\frac{1}{2}}(kh)^2$, $q \approx 3g/4h$. The condition (5.16) reduces to

$$\mathscr{U} \equiv a/(k^2 h^3) < (2/3)^{\frac{1}{2}}$$

which is the criterion for the Ursell parameter ($\mathscr{U}$) to be small enough for dispersive wave theories (Stokes' waves, KdV) to be valid, hence for the existence of a non-breaking wave train.

Following the arguments used here, (5.10) can also be derived from (16.94–16.95) in Whitham (1974) which describe the interaction of the waves with the mean flow. A review of wave–mean flow interaction is given by Grimshaw (1984). Whitham obtains the above equations (5.1) and (5.2) through linearization of the wave–mean flow interaction problem, assuming that the mean flow is smaller than the wave motion. For large Ursell parameter, the mean flow interacts resonantly with the waves and is comparable to the wave motion. We have treated this range by taking into account quadratic terms in U and ξ_{10} and obtained (5.10). It was seen that the effect of the mean flow on the propagation of the wave group (Doppler shift) plays a crucial role in the theory obtained. Figure 3 shows the dependence of ξ_{10} on the wave amplitude, a, from (5.12) as compared to the value given by (5.2) ($kh = 1$). The values are close even for fairly large Ursell parameter values for which, however, linearization is not valid. Hence, the present theory should be used to justify the results. The existence of very large 'surf-beats' in shallow water is confirmed by numerous observations, e.g. Guza & Thornton (1982).

In the following section we suggest an application of the results in §§2 and 3 to the interaction of many waves.

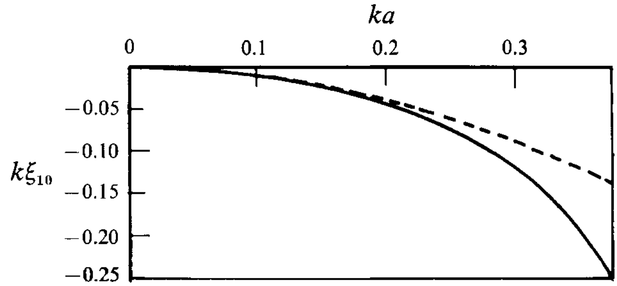

FIGURE 3. The dependence of the normalized setdown, $k\xi_{10}$, on the normalized wave amplitude, ka. The 'linearized' result (----) is compared to the present model (——); $kh = 1$.

6. Forced waves near resonance

In this section, we wish to examine the implications of resonance of the locked waves and suggest a way to modify the numerical scheme of Stiassnie & Shemer (1987) using the ideas of §§2 and 3. We start from (3.1) and (3.2). When $\mu_1 = O(1)$, (3.2) can be integrated with respect to the fast time to yield

$$B_0'' = -T_{0,1,2,3} B_1^* B_2 B_3 \frac{e^{i\mu_1 t}}{\mu_1}, \tag{6.1}$$

the wave forced by B_1, B_2 and B_3. We see that if $\mu_1 = O(1)$, then

$$B_0'' = B_0''(t) = O(\epsilon^3), \tag{6.2}$$

but if $\mu_1 = O(\epsilon)$

$$B_0'' = B_0''(t_1) = O(\epsilon^2). \tag{6.3}$$

This last possibility is not accounted for in previous work. Of course, the distinction among the different orders cannot be clear cut, in particular when we wish to perform computations.

This is not a problem in the transition between $\mu_1 = O(1)$ to $\mu_1 = O(\epsilon)$, since both give rise to a forced wave. The transition to $O(\epsilon^2)$, however, is not smooth. In a computation scheme, a certain number of free wave components is chosen (e.g. five in Stiassnie & Shemer 1987) and the rest of the waves are taken to be forced waves.

If we have a situation in which $\mu_1 \gtrsim \epsilon^2$, use of (6.1) will give

$$B'' = B''(t_2) \lesssim \epsilon, \tag{6.4}$$

which is near the order of the free waves. This upsets the hierarchy of orders. Also, the integration leading to (6.1) which relies on timescale separation is less justified since the phase $e^{i\mu_1 t}$ is varying on a timescale close to t_2, over which B_1, B_2 and B_3 vary.

A similar situation is encountered in the interaction of gravity–capillary waves. Here trio resonance may occur and the interaction is governed by (2.2). Similar to (3.1), the wave is written as

$$[B(\boldsymbol{k}, t_1) + B'(\boldsymbol{k}, t, t_1) + \ldots]\, e^{-i\omega(k)t}, \tag{6.5}$$

where $B = O(\epsilon), \quad B' = O(\epsilon^2).$

The problem of scale separation arises when

$$\mu \equiv \omega_0 - (\omega_1 \pm \omega_2); \quad \omega_0 = \omega(\boldsymbol{k}_1 \pm \boldsymbol{k}_2)$$

is in the range $\mu \gtrsim \epsilon$.

We now recall that in §2 we obtained an equation for the amplitude of the forced wave which was valid for all magnitudes of μ. If we account for the phases, we can replace (2.8) by

$$\left(\frac{B_2^*}{B_1}+\frac{B_1^*}{B_2}\right)B_0^2-\frac{\mu}{2V}B_0-B_1B_2=0 \tag{6.6}$$

for the sum interaction, and a similar equation for the difference wave. We propose this equation for approximating $B_{1,2}$, the wave forced by B_1 and B_2, in the vicinity of resonance. Similarly, in the case of quartet interaction, we write for three free waves B_1, B_2, B_3 and a forced wave B_0, where $\boldsymbol{k}_0=\boldsymbol{k}_2+\boldsymbol{k}_1-\boldsymbol{k}_3$; $\omega_0=\omega(\boldsymbol{k}_0)$, the following equations, derived from (3.2):

$$(\Omega_p-\omega_p')B_p=2T_{p,q,r,s}B_q^*B_rB_s;\quad p=0,1,2,3;\quad p+q=r+s=3, \tag{6.7}$$

$$\Omega_0+\Omega_3=\Omega_2+\Omega_1, \tag{6.8}$$

where

$$\omega_p'=\omega_p+T_p|B_p|^2+2\sum_{q\neq p}T_{pq}|B_q|^2$$

are the corrected frequencies, as in (3.4). Again we get a quadratic equation for B_0:

$$\left[-\left(\frac{B_1B_2}{B_3}\right)^*+\frac{B_1^*B_3}{B_2}+\frac{B_2^*B_3}{B_1}\right]B_0^2-\frac{\mu_1}{2T}B_0-B_1B_2B_3^*=0. \tag{6.9}$$

μ_1, the detuning is given by

$$\mu_1=\omega_2'+\omega_3'-\omega_1'-\omega_0'.$$

Similar equations are obtained for the other quartet, and quintet, interaction forced waves. The solutions offer a smooth transition as the value of μ (or μ_1) decreases and resonance is approached.

Equation (6.6) or (6.9) yields the value of the wave forced by two or three waves, which are members of a nearly resonant trio or quartet, respectively. This is based on the assumption that the waves are in a steady state. This requires (unless there is an exact phase match of the forced wave) that the forced wave B_0 will be smaller than the $O(\epsilon)$ waves B_1, B_2 and B_3. From (6.9), we see that $B_0/B_1=O(\epsilon^2/\mu_1)$. We may find the accuracy attained in (6.9). By using the above ratio of B_0/B_1 in (3.2), we find that the timescale for the evolution of B_0 is μ_1^{-1} (where the wave period is $O(1)$). The corresponding evolution equation for, say, B_1, gives a timescale of μ_1/ϵ^2. The ratio of these timescales is ϵ^4/μ_1^2. Thus, the relative error in the approximation involving (6.9) is $O(\epsilon^4/\mu_1^2)$. We have assumed that $\mu_1>\epsilon^2$. If we take, say, $\mu_1\approx\epsilon^{\frac{3}{2}}$, we get a relative error which is $O(\epsilon)$. In the case of trio interaction, we find from (6.6) that the relative error is $O(\epsilon^2/\mu^2)$.

We have seen that as long as the waves are not too close to resonance, we may treat the additional wave as a forced wave by using (6.6) or (6.9). An example of the need for the correction discussed here is seen in figure 1. As noted in §3, there is a discontinuity in the curve for $\hat{T}_{1,2}$, the corrected interaction coefficient for waves with wavenumbers k_1 and k_2, near $k_2/k_1=0.4$. This corresponds to a wavelength $2\pi/k_2=2.44$ cm for which wave 2 is in resonance with its second harmonic. This leads in a naive calculation of $\hat{T}_{1,2}$ to infinite values. After correcting the forced wave contribution due to that Wilton ripple, this singularity is removed, leaving only the discontinuity seen in figure 1.

The above method has implications on the free-wave evolution as well. The kernel T has terms with μ in the denominator, the kernel for quintet interaction has terms with μ or μ_1 in the denominator. We saw in §3 (equation (3.6)) that these are due to forced waves. We have also seen that the form involving the linear frequencies, (3.5), which is the expression normally used in the Zakharov-type equations, gives values which are too high as resonance is approached. This can be corrected by replacing the corresponding term by a term obtained using the solution of (6.6), say, in the quartet interaction equation, as done in (3.7). This means, for example, in the case of $T'_{1,2}$, replacing it by

$$\hat{T}_{1,2} = \frac{-2V}{\omega_3 - \Omega_3} + T'_{1,2},$$

where $\Omega_3 - \omega_3 = y - \mu$ is found in the manner of (3.8) (it is sufficient to use the simplified, trio interaction, form derivable from (6.6) in the manner of §2). This is important when calculating class I and class II interactions (as in Stiassnie & Shemer 1987) or when calculating quartet interaction for gravity–capillary waves. We note that for just five free waves, Stiassnie & Shemer had to consider 3700 locked waves. This number is much too large to enable treatment of the full system. In order to account for approach to resonance, the above approximation can be used.

7. Conclusion

There is a continuous increase in the magnitude of a wave forced by a number of free waves, as resonance is approached. We have illustrated this for the example of trio interaction. The growth of the forced wave is accompanied by an increasing shift in the free-wave frequencies. This detuning, in turn, limits the growth of the forced wave. We have obtained new expressions for the shift of the phase speed of one wave train in the presence of another, for Wilton ripples and for the wave setdown and drift current, which are uniformly valid as resonance is approached. The ideas were applied to derive uniformly valid kernels for the Zakharov equation and for its modifications.

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Appendix

The interaction terms for the Zakharov equation kernel $T(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ were given by HGS for gravity–capillary waves in water of infinite depth. We give here the form of the kernel for waves in water of finite depth.

Denote $T(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = T_{0,1,2,3}$, then

$$T_{0,1,2,3} = -\frac{2V^{(-)}_{3,3-1,1}\, 2V^{(-)}_{0,2,0-2}}{\omega_{1-3} - \omega_3 + \omega_1} - \frac{2V^{(-)}_{2,0,2-0}\, V^{(-)}_{1,1-3,3}}{\omega_{1-3} - \omega_1 + \omega_3} - \frac{2V^{(-)}_{2,2-1,1}\, V^{(-)}_{0,3,0-3}}{\omega_{1-2} - \omega_2 + \omega_1}$$
$$- \frac{2V^{(-)}_{3,0,3-0}\, V^{(-)}_{1,1-2,2}}{\omega_{1-2} - \omega_1 + \omega_2} - \frac{2V^{(-)}_{0+1,0,1}\, V^{(-)}_{2+3,2,3}}{\omega_{2+3} - \omega_2 - \omega_3} - \frac{2V^{(+)}_{-2-3,2,3}\, V^{(+)}_{0,1,-0-1}}{\omega_{2+3} + \omega_2 + \omega_3} + W'_{0,1,2,3}.$$

The second-order interaction coefficients are given by

$$V^{(\pm)}_{0,1,2} = \frac{1}{8\pi\sqrt{2}} \left\{ [\boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \pm \sigma_0 \sigma_1] \left[\frac{\omega_0 \omega_1}{\omega_2} \frac{\sigma_2}{\sigma_0 \sigma_1} \right]^{\frac{1}{2}} + [\boldsymbol{k}_0 \cdot \boldsymbol{k}_2 \pm \sigma_0 \sigma_2] \left[\frac{\omega_0 \omega_2}{\omega_1} \frac{\sigma_1}{\sigma_0 \omega_2} \right]^{\frac{1}{2}} \right.$$
$$\left. + [\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \sigma_1 \sigma_2] \left[\frac{\omega_1 \omega_2}{\omega_0} \frac{\sigma_0}{\sigma_1 \sigma_2} \right]^{\frac{1}{2}} \right\},$$

where $\sigma_j = |\boldsymbol{k}_j| \tanh(|\boldsymbol{k}_j| h)$ and $\omega_j^2 = g|\boldsymbol{k}_j| + S_0 |\boldsymbol{k}_j|^3$; $j = 0, 1, 2, 3$.

The third-order interaction coefficient

$$W'_{0,1,2,3} = W'(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$$

is given by

$$W'_{0,1,2,3} = W_{0,1,2,3} - \frac{S_0}{32\pi^2}\left(\frac{\sigma_0\sigma_1\sigma_2\sigma_3}{\omega_0\omega_1\omega_2\omega_3}\right)^{\frac{1}{2}} [(\boldsymbol{k}_0\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3) + (\boldsymbol{k}_0\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3) + (\boldsymbol{k}_0\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)],$$

where

$$W_{0,1,2,3} = \overline{W}_{-0,-1,2,3} + \overline{W}_{2,3,-0,-1} - \overline{W}_{2,-1,-0,3} - \overline{W}_{-0,2,-1,3} - \overline{W}_{-0,3,2,-1} - \overline{W}_{3,-1,2,-0}$$

with

$$\overline{W}_{0,1,2,3} = \frac{1}{64\pi^2}\left[\frac{\omega_0\omega_1}{\omega_2\omega_3}\sigma_0\sigma_1\sigma_2\sigma_3\right]^{\frac{1}{2}} \{2(\sigma_1+\sigma_2) - \sigma_{1+3} - \sigma_{1+2} - \sigma_{0+3} - \sigma_{0+2}\}$$

and

$$\sigma_{i\pm j} = \sigma(\boldsymbol{k}_i \pm \boldsymbol{k}_j), \quad \omega_{i\pm j} = \omega(\boldsymbol{k}_i \pm \boldsymbol{k}_j); \quad i, j = 0, 1, 2, 3.$$

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